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# STABILIZATION OF CONTROLLED SYSTEMS WITH RESPECT TO A PART OF THE VARIABLES

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We use the method of Liapunov functions to obtain a set of control laws securing the Liapunov stability of an unperturbed motion of the controlled system and its asymptotic stability with respect to a part of the variables. The properties of the stabilizing laws obtained here are determined and the laws which are compulsorily optimal are separated from the set. An example is given.

**1. Statement of the problem.** We consider a controlled system, the perturbed motion of which on the set $H$ is described by the equation

$$x^{\cdot} = \Phi(x, u, t), \ x \in R^n, \ u \in R^m, \ m \leqslant n \tag{1.1}$$
$$H = \{x, t\colon \|x\| < h, \ t \geqslant 0\} \ (\|x\|^2 = x^T x, \ h = \text{const} > 0)$$

Here $\Phi(x, u, t) = \{\Phi_1, \ldots, \Phi_n\}$ is a vector function satisfying the conditions of existence and uniqueness of the solution of (1.1) on a certain set of continuous control laws

$$u = u(x, t), \quad u(0, t) = 0 \tag{1.2}$$

Following [1 — 3], we formulate the problem of stabilization in the following manner.

Problem. Let a set of control laws (1.2) be given. We require to separate from (1.2) a subset of laws on which the unperturbed motion $x = 0$ of the system (1.1) is Liapunov stable, and asymptotically stable with respect to the variables $x_1, \ldots, x_r$, $r \leqslant n$.

The set of laws emerging as the result of solving the problem, will be called the set of stabilizing control laws or the set of stabilization laws.

**2. Stabilization theorem.** In order to solve the problem, we introduce the scalar functions $V(x, t)$ and $W(x, t)$ which are positive-definite on the set $H_1 = \{x, t\colon \|x\| < h_1 \leqslant h, \ t \geqslant 0\}$, $V(x, t)$ with respect to the variables $x_1, \ldots, x_n$ and $W(x, t)$ with respect to $x_1, \ldots, x_r$.

In analogy with [3, 4], the function $W(x, t)$ is called positive-definite with respect to the variables $x_1, \ldots, x_r$ on $H_1$, if $W(0, t) \equiv 0$ and a continuous, positive-definite Liapunov function $w(x_1, \ldots, x_r)$ can be shown on the set $H_0 = \{x_1, \ldots, x_r\colon x_1^2 + \ldots + x_r^2 < h_1^2\}$ such that for all $x, t \in H_1$

$$W(x, t) \geqslant w(x_1, \ldots, x_r) \tag{2.1}$$

Let the function $V(x, t)$ be defined and continuous on $H_1$ together with its partial derivatives $V_x' = V_{x_1}', \ldots, V_{x_n}']^T$ and $V_t = dV / dt$.

Theorem. Any control law (1.2) which ensures that the condition

$$V_x^T \Phi(x, y, t) + V_t = -W(x, t) \tag{2.2}$$

holds on the set $H_1$, is a stabilizing control law for the system (1.1) provided that for all $x, t \in H_1$

$$x_1\Phi_1(x, u, t) + \ldots + x_r\Phi_r(x, u, t) \leqslant N, \quad N > 0 \tag{2.3}$$

To prove the theorem we first note that in accordance with the first Liapunov theorem and the condition (2.2) holding, the closed system (1.1),(1.2) is stable with respect to the variables $x_1, \ldots, x_n$ and for any arbitrarily small $\varepsilon > 0$, a number $\delta > 0$ can be shown such that $\| x(t) \| < \varepsilon$ for all $t > 0$ provided that

$$\| x(0) \| \leqslant \delta \tag{2.4}$$

Moreover, by virtue of the inequality (2.1) we find that for any motion originating in the region (2.4) we have

$$\int_0^t w(x_1(\tau), \ldots, x_r(\tau))\, d\tau \leqslant V(x(0), 0) - V(x(t), t), \quad t > 0$$

and, since $V^{\cdot}(x, t) \leqslant 0$ on $H_1$, we have

$$\int_0^\infty w(x_1(t), \ldots, x_r(t))\, dt \leqslant V(x(0), 0) \tag{2.5}$$

Next we shall show that (2.5) and the condition (2.3) of the theorem together yield

$$[x^2_1(t) + \ldots + x^2_r(t)] \to 0 \qquad \text{as} \quad t \to \infty \tag{2.6}$$

In fact, let us assume that (2.6) is incorrect. Then, using the very concept of the limit we can show an infinitely increasing sequence of time instances $\{t_k\}$ such that

$$x^2_1(t_k) + \ldots + x^2_r(t_k) \geqslant s_1, \; s_1 = \text{const} > 0, \; k = 1, 2, \ldots \tag{2.7}$$

From (2.7) and (2.3) follows

$$x^2_1(t) + \ldots, x_r^2(t) \geqslant s_2, \; s_2 = \text{const} > 0, \; s_2 < s_1$$

for at least all $t \in [t_k - \Delta t, \; t_k]$, where $\Delta t = (s_1 - s_2) / 2N$. Let now

$$\eta = \min\Big(w(x_1, \ldots, x_r) \quad \text{for} \quad s_2 \leqslant \sum_{i=1}^r x_i^2 < \varepsilon^2\Big)$$

Then for any $t_k$ we have

$$\int_{t_k - \Delta t}^{t_k} w(x_1(t), \ldots, x_r(t))\, dt \geqslant \eta \Delta t$$

which contradicts the convergence of the integral (2.5). Consequently, it follows that (2.6) holds and the closed system (1.1),(1.2) is asymptotically stable with respect to the variables $x_1, \ldots, x_r$.

We note that the conditions of the theorem do not include the condition of an infinitesimal upper bound not only with respect to the variables $x_1, \ldots, x_r$, but also with respect to $x_1, \ldots, x_n$. This enables us to assert, in particular, that the control law $u = u^0(x, t)$ satisfying the conditions of the theorem given above, is optimal with respect to the functional

$$J(u) = \int_0^\infty L(x, u, t)\, dt, \qquad L(x, u, t) \geqslant 0$$

provided that $L(x, u^\circ(x, t), t) = W(x, t)$ and the relation

$$V_x{}^T \Phi(x, u, t) + V_t + L(x, u, t) \geqslant 0$$

holds for any vector $u \neq u^\circ$.

The validity of the above assertion follows directly from the fundamental theorem of optimal stabilization [1] the conditions of which allow the omission of the requirement of the infinitesimal upper bound. We note that

$$\min_u \int_0^\infty L(x, u, t)\, dt \leqslant V(x(0), 0) \tag{2.8}$$

and the equals sign appears in (2.8) only when the unperturbed motion of the system

$$x^\cdot = \Phi(x, u^\circ(x, t), t)$$

is asymptotically stable with respect to $x_1, \ldots, x_n$.

**3. Particular cases of stabilization.** Let a perturbed motion of the system stabilized with respect to the variables $x_1, \ldots, x_r$ be described by the equation

$$x^\cdot = F(x, t) + G(x, t)\, u \tag{3.1}$$

where $F$ is an $n$-dimensional vector function and $G$ is a ( $n \times m$ )-matrix. The condition (2.2) is reduced in the present case to the form

$$V_x^T (F + Gu) + V_t = -W(x, t) \tag{3.2}$$

and the set of laws ensuring that (3.2) holds is determined by the expression

$$u = p(x, t) - (W(x, t) + V_x^T F + V_t)(V_x^T G G^T V_x)^{-1} G^T V_x$$

provided that

$$p^T G^T V_x = 0, \qquad V_x^T G G^T V_x \neq 0$$

on $H_1$

In the general case the function $V_x^T G G^T V_x \neq 0$ may vanish when $\| x \| \neq 0$, therefore the solution of the problem of stabilizing the system (3.1) is naturally sought in the form

$$u = l(x, t) - \lambda G^T V_x \tag{3.3}$$

where $l$ and $\lambda$ are arbitrary vector and scalar ($\lambda = \lambda(x, t)$) functions connected by the equation

$$V_t = -W(x, t) - V_x^T (F + Gl) + \lambda V_x^T G G^T V_x \tag{3.4}$$

and by the restriction

$$[x_1, \ldots, x_r, 0, \ldots, 0]\, (F + Gl - \lambda G G^T V_x) \leqslant N, \quad x, t \in H_1$$

Setting in (3.3) and (3.4) $l \equiv 0$ and requiring that the inequality

$$\lambda(x, t) > 0 \tag{3.5}$$

holds on $H_1$, we obtain the following stabilization law:

$$u = -\lambda G^T V_x, \quad V_t = -W(x, t) - V_x^T F + \lambda V_x^T G G^T V_x$$

which is compulsorily optimal [5].

We note that when the inequality (3.5) holds, any of the laws (3.3) acting upon the motions (3.1) of the system produces a minimum in the value of the functional

$$J = \int_0^\infty \left[ W(x, t) + \frac{1}{2\lambda} \| l \|^2 - \frac{\lambda}{2} \| G^T V_x \|^2 - \frac{1}{\lambda} u^T l + \frac{1}{2\lambda} \| u \|^2 \right] dt$$

The intermediate position between the stabilization laws (3.3) and (3.6) is occupied by

$$u = -\lambda G^T V_x + P G^T V_x$$

where $P = P(x, t)$ is an arbitrary skew-symmetric matrix and

$$u = -\Lambda\, G^T V_x, \quad V_t = -W(x, t) - V_x^T F + V_x^T G \Lambda\, G^T V_x \tag{3.7}$$

where $\Lambda = \Lambda(x, t)$ is a positive-definite ( $m \times m$ )-matrix. Incidentally, in accordance with [5], the control laws of the type (3.7) can be regarded as compulsorily optimal laws, provided that the quadratic form $2zw = u^T \Lambda^{-1} u$ is adopted as the forcing norm.

It is clear that the control law (3.6) guarantees the stabilization of the system

$$x^\cdot = F(x, t) + G(x, t)\,\varphi(u, t) \qquad (3.8)$$

with respect to the variables $x_1, \ldots, x_r$ for any vector functions $\varphi(u, t)$ satisfying the condition

$$u^T \varphi(u, t) \geqslant u^T u, \qquad t \geqslant 0$$

and a constraint of the form (2.3), and the law (3.7) in the case when the matrix $\Lambda$ is diagonal for any $\varphi(u, t) = \{\varphi_1(u_1, t), \ldots, \varphi_m(u_m, t)\}$ satisfying the inequalities

$$u_j \varphi_j(u_j, t) \geqslant u_j^2, \qquad t \geqslant 0 \quad (j = 1, \ldots, m)$$

In addition, if on the motions (3.1), (3.6) we have

$$\min_u I(u) = \varkappa^\circ, \; I(u) = \int_0^\infty \left[W(x, t) - \frac{\lambda}{2} V_x^T G G^T V_x + \frac{1}{2\lambda} u^T u\right] dt$$

then on the motions (3.8), (3.6) we have

$$\varkappa^\circ = \max_\varphi \min_u I(u).$$

Similar relations take place in the case (3.7) when

$$I(u) = \int_0^\infty \left[W(x, t) - \frac{1}{2} V_x^T G \Lambda G^T V_x + \frac{1}{2} u^T \Lambda^{-1} u\right] dt$$

**4. Example.** We consider the problem of damping the rotation of a body clamped at one point. Let $r = 2$, and the perturbed motion is described by the equations

$$x_1^\cdot = \frac{B-C}{A} x_2 x_3 + \frac{1}{A} \varphi_1(u_1, t), \quad x_2^\cdot = \frac{C-B}{B} x_3 x_1 + \frac{1}{B} \varphi_2(u_2, t)$$

$$x_3^\cdot = \frac{A-B}{C} x_1 x_2$$

where $A$, $B$ and $C$ are principal moments of inertia of the body. In this case (3.7) can be satisfied by setting

$$V = A x_1^2 + B x_2^2 + C x_3^2, \quad W = 4\lambda_1 x_1^2 + 4\lambda_2 x_2^2, \quad u_1 = -2\lambda_1 x_1$$
$$u_2 = -2\lambda_2 x_2$$

where $\lambda_1$ and $\lambda_2$ are positive constants.

We note that the stabilization of the system

$$x_1^\cdot = \frac{B-C}{A} x_2 x_3 + \frac{1}{A} u_1, \quad x_2^\cdot = \frac{C-A}{B} x_3 x_1 + \frac{1}{B} u_2, \quad x_3^\cdot = \frac{A-B}{C} x_1 x_2$$

with respect to the variables $x_1$ and $x_2$ is guaranteed by the control laws obtained with the functional

$$J(u_1, u_2) = \int_0^\infty \left[2(\lambda_1 x_1^2 + \lambda_2 x_2^2) + \frac{1}{2}\left(\frac{1}{\lambda_1} u_1^2 + \frac{1}{\lambda_2} u_2^2\right)\right] dt$$

assuming a minimum value.

The above example can be generalized by making the following assertion. Let the stabilized system be described by the equation

$$x^\cdot = F(x) + G(x) u$$

and let $V(x)$ be a positive-definite Liapunov function satisfying the condition $V_x^T F \equiv 0$. The control law

$$u = -\Lambda G^T V_x$$

quarantees the asymptotic stability of the system with respect to the variables $y_1 = \Psi_1(x), \ldots, y_r = \Psi_r(x)$ with the functional

$$J(u) = \frac{1}{2}\int_0^\infty [V_x^T G\Lambda G^T V_x + u^T \Lambda^{-1} u]\, dt$$

assuming its minimum value, provided that the function $V_x^T G\Lambda G^T V_x$ is positive-definite with respect to $y_1, \ldots, y_r$.

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# LAMINAR AXISYMMETRIC JET SUBMERGED IN A ROTATING FLUID

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A solution for a weakly nonself-similar axisymmetric jet submerged in a rotating viscous incompressible fluid is derived in a boundary layer approximation. An asymptotic expression is obtained for the jet field at considerable distances from the source, where it becomes self-similar.

**1.** Let a half-space filled by a viscous incompressible fluid and its solid plane boundary rotate at constant angular velocity $\omega$ around an axis normal to that plane.

We attach to the solid plane a right-hand system of cylindrical coordinates $r, \varphi, z$ and make the half-space boundary to coincide with the plane $z = 0$ so that for every point of the fluid $z > 0$. Let us consider the problem of slow steady axisymmetric relative motions of the fluid in the half-space, induced by the velocity distribution at the solid plane

$$\mathbf{v}|_{z=0} = \mathbf{e}_z w_0(r) \tag{1.1}$$

with conditions at infinity